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Numerical Approximation of Eigenvalues of Sturm–Liouville Systems

B. A. HARGRAVE

Department of Mathematics, King's College, University of Aberdeen, Aberdeen, Scotland

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A new technique based on the Prüfer transformations is proposed. This technique is illustrated using several examples of Sturm–Liouville systems offering varying degrees of computational difficulty. For some of the examples, other techniques have been used to calculate eigenvalues, and in these cases, results obtained are compared. The technique appears capable of generalization to multiparameter systems of differential equations.

1. INTRODUCTION

Sturm–Liouville systems arise as the mathematical formulation of boundary value problems. Typical examples are a vibrating string with elastically held ends and vibrating elliptic membrane clamped at the edge. The former problem may be formulated immediately as a Sturm–Liouville problem, while the latter is a two-dimensional problem, which may be reduced to two ordinary differential equations, one of which is associated with a Sturm–Liouville system.

The technique, known as separation of variables, used in the latter problem, may also be used for problems in higher dimensions. If one attempts to solve an n-dimensional problem, then one would expect to obtain n-differential equations with eigenvalues occurring as $(n-1)$ separation constants. A more general problem of this type, the multiparameter problem for ordinary differential equations, has been considered from a theoretical viewpoint by several authors (for example [2, 11]) while eigenvalues have been estimated using a numerical technique [8]. However the numerical approach has only been successful for equations involving two parameters. When one proceeds beyond two parameters, the existing techniques do not appear to be useful.

It is in this context that an alternative method for efficient estimation of eigenvalues of Sturm–Liouville systems is proposed. However, the technique described here as applications to Sturm–Liouville systems as it appears to be an efficient way of treating the difficult problem of determining the eigenvalues of a system, which involves a large parameter and a turning-point. It also appears to be useful

for problems involving moderately large parameters where asymptotic series may require many terms in order to be of the required degree of accuracy.

To illustrate the range of applicability of the methods, eight tables of eigenvalues are given. In these tables, the computation times quoted are those taken by the I.C.L. System 4/70 computer at the University of Aberdeen. The results appearing in these tables were obtained using a sixth-order Runge–Kutta process.

2. Sturm–Liouville Systems

The general form of a Sturm–Liouville system may be written as

$$\begin{gathered} \frac{d}{dx}\left(p(x)\frac{dw}{dx}\right) + \{\lambda\rho(x) - q(x)\}\, w(x) = 0, \\ \sin \alpha w'(a) - \cos \alpha w(a) = \sin \beta w'(b) - \cos \beta w(b) = 0. \end{gathered} \tag{2.1}$$

If the quantities in (2.1) are subject to the following restrictions:

(i) $[a, b]$ is a finite real interval,

(ii) ρ and q are continuous and p is continuously differentiable on $[a, b]$,

(iii) $p(x)$ and $\rho(x)$ are positive, $\forall x \in [a, b]$,

(iv) $0 \leqslant \alpha < \pi$, $0 < \beta \leqslant \pi$,

then (2.1) is a regular Sturm–Liouville system.

The theory of eigenfunctions and eigenvalues of regular Sturm–Liouville systems is well known [9, p. 373]. The following results for such systems are required:

(a) There exists a unique value of λ such that the system (2.1) admits a solution having a prescribed number of zeros in (a, b).

(b) If the set of eigenvalues is denoted by $\{\lambda_i\}$, the suffix denoting the number of zeros in (a, b) of the corresponding eigenfunction, then $j > i$ implies that $\lambda_j > \lambda_i$.

(c) Let $q = \min\{q(x): x \in [a, b]\}$, and $Q = \max\{q(x): x \in [a, b]\}$.

Let the system (2.1) with $q(x)$ replaced by q have eigenvalues μ_i and the system (2.1) with $q(x)$ replaced by Q have eigenvalues η_i. Then

$$\mu_i \leqslant \lambda_i \leqslant \eta_i .$$

It is not necessary to consider the most general form of the Sturm–Liouville system. We shall use the following form of (2.1),

$$\begin{gathered} w'' + \{\lambda\rho(x) - q(x)\}\, w(x) = 0, \\ \sin \alpha w'(a) - \cos \alpha w(a) = \sin \beta w'(b) - \cos \beta w(b) = 0. \end{gathered} \tag{2.2}$$

The differential equation in (2.2) is of similar form to one of the n equations in the multiparameter problem.

There is no loss of generality in considering (2.2) since, for regular Sturm–Liouville systems, (2.1) may be reduced to the even simpler form.

$$(d^2u/dt^2) + \{\lambda - \hat{q}(t)\}\, u = 0, \tag{2.3}$$

with associated boundary conditions, by the Liouville substitution [9, p. 340]

$$w = u/(p(x)\, \rho(x))^{1/4}, \qquad t = \int (\rho(x)/p(x))^{1/2}\, dx.$$

The eigenvalues of (2.3) are identical to those of (2.1).

3. Prüfer Substitutions

There are classically two variants of the Prüfer substitution. The purpose of the substitutions is to transform the second-order equation

$$w'' + Q(x)\, w(x) = 0$$

to two first-order equations, one of which is independent of the solution of the other equation. The eigenvalue may then be obtained from this first-order equation.

To apply the Prüfer substitution, introduce $r(x)$, $\theta(x)$ such that

$$w'(x) = r(x) \cos \theta(x), \qquad w(x) = r(x) \sin \theta(x).$$

The resulting first-order differential equations obtained from the original problem are

$$d\theta/dx = Q(x) \sin^2 \theta + \cos^2 \theta, \tag{3.1}$$

$$dr/dx = (1/2)[1 - Q(x)]\, r \sin 2\theta. \tag{3.2}$$

For a regular Sturm–Liouville system, a typical set of solutions, for various values of λ, of (3.1) with initial condition

$$\theta(a) = \alpha$$

would be as shown in Fig. 1.

The boundary conditions of (2.2) determine the values of θ at the end points, i.e.,

$$\theta(a) = \alpha, \qquad \theta(b) = \beta + m\pi. \tag{3.3}$$

These boundary conditions for θ ensure that the solution of the problem (2.2) has m zeros in (a, b).

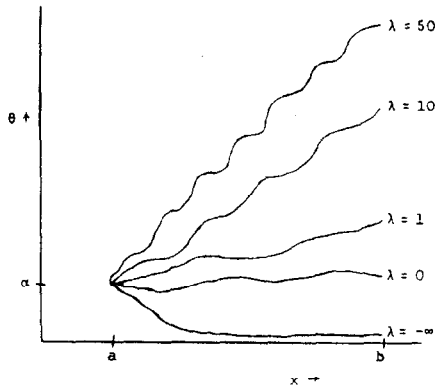

FIGURE 1

While the Prüfer method offers a simple way of obtaining eigenvalues [3], it is not always the most efficient way. In particular, if $Q(x)$ is large the truncation error occurring in any method of solution for ordinary differential equations may be large and the results consequently not as accurate as the order of the method indicates.

A method that is useful for large values of $Q(x)$ is the modified Prüfer method [9, p. 402], which has been used by Fix [6] in his analytic investigation of eigenvalues of differential equations corresponding to large numbers of zeros. Introducing $R(x)$, $\phi(x)$ defined by,

$$w'(x) = R(x)\{Q(x)\}^{1/4} \cos \phi, \qquad w(x) = (R(x)/\{Q(x)\}^{1/4}) \sin \phi \quad (3.4a,b)$$

leads to

$$d\phi/dx = [Q(x)]^{1/2} + (Q'/4Q) \sin 2\phi, \tag{3.5}$$

and

$$dR/dx = -(RQ'/4Q) \cos 2\phi. \tag{3.6}$$

The boundary conditions are not immediately obvious for the modified Prüfer equation, but they may be obtained by investigating the relation between Prüfer and modified Prüfer variables as defined above. Observe that one may only apply the modified Prüfer method in $[a, b]$ if $Q(x) > 0$, $\forall x \in [a, b]$. Comparing the definitions of the respective variables in the substitutions

$$r(x) \cos \theta(x) = R(x)\{Q(x)\}^{1/4} \cos \phi(x),$$
$$r(x) \sin \theta(x) = R(x)\{Q(x)\}^{-1/4} \sin \phi(x),$$

so that

$$\tan \theta(x) = \tan \phi(x)/[Q(x)]^{1/2}. \tag{3.7}$$

Thus, the zeros and singularities of $\tan \theta(x)$ and $\tan \phi(x)$ occur for the same values of x, and we may deduce that

$$|\theta(x) - \phi(x)| < \pi/2.$$

Transforming (3.3) using (3.7), leads to

$$\begin{aligned} \tilde{\alpha} &= \tan^{-1}\{\tan \alpha[Q(a)]^{1/2}\}, && \text{if } \alpha \leqslant \pi/2, \\ &= \pi + \tan^{-1}\{\tan \alpha[Q(a)]^{1/2}\}, && \text{if } \alpha > \pi/2, \end{aligned} \tag{3.8a}$$

and

$$\begin{aligned} \tilde{\beta} &= m\pi + \tan^{-1}\{\tan \beta[Q(b)]^{1/2}\}, && \text{if } \beta \leqslant \pi/2, \\ &= (m+1)\pi + \tan^{-1}\{\tan \beta[Q(b)]^{1/2}\}, && \text{if } \beta > \pi/2, \end{aligned} \tag{3.8b}$$

where

$$-\pi/2 < \tan^{-1} x \leqslant \pi/2.$$

Equation (3.5) used in conjunction with Eqs. (3.8a, b) produces a powerful method for the solution of problems for which $Q(x)$ is bounded below by a positive constant. For highly oscillatory solutions, the positive constant is large and Eq. (3.5) leads to steadily increasing solutions, which do not suffer from the usual inaccuracies that affect methods for highly oscillatory solutions of Sturm–Liouville systems.

4. Details of the Numerical Technique

As Eq. (3.7) facilitates the transformation from Prüfer to modified Prüfer phase angles, provided, of course, that one does not attempt to transform when $Q(x) = 0$, it is possible to illustrate the numerical method using the Prüfer phase angle only.

The regular Sturm–Liouville system (2.2) may be cast as the initial value problem

$$\begin{aligned} \theta'(x) &= \cos^2 \theta + \{\lambda\rho(x) - q(x)\} \sin^2 \theta, \\ \theta(a) &= \alpha. \end{aligned} \tag{4.1}$$

The right-hand boundary condition has yet to be imposed on this system. For a fixed value of λ, it is possible to solve (4.1) as an initial value problem using one of many known techniques for this type of differential equation, to obtain the value of θ at b. This value will clearly depend on both b and λ so that we denote it by $\theta(b; \lambda)$. Hence, we can construct a function $f\colon \mathbb{R} \to \mathbb{R}_+$ defined by $f(\lambda) = \theta(b; \lambda)$. For a given value of β, we can now define the function f_0 by

$$f_0(\lambda) = f(\lambda) - \beta. \tag{4.2}$$

If we are able to find a value of λ such that

$$f_0(\lambda) = 0, \tag{4.3}$$

then this is the eigenvalue of the system (2.2). The value of λ is unique by result (a) for regular Sturm–Liouville systems and in fact f_0 is an increasing function of λ. For higher eigenvalues we have only to investigate the unique zero of each of the functions f_m defined by

$$f_m(\lambda) = f(\lambda) - (\beta + m\pi). \tag{4.4}$$

The techniques used for the determination of λ are iterative. From property (c) of Section 2, a priori bounds are known for each eigenvalue so that Müller's method [5, p. 75] is particularly suitable for this type of equation.

Earlier in this section, techniques for initial value problems were mentioned. A comprehensive description of each type of method and a comparison of these methods may be found in [10]. A sixth-order Runge–Kutta process was found to be more efficient than other methods when applied to the problems in Sections 5–7. By using the modified Prüfer substitution, it is possible to treat the case of large positive $\{\lambda\rho(x) - q(x)\}$, provided that the derivative of this function is not appreciably larger than the function itself. However, if $\{\lambda\rho(x) - q((x)\}$ is large and negative, one uses the Prüfer method (an analog of the modified Prüfer equation may be formulated, but this involves a $\sin h\, 2\phi$ that dominates the right-hand side).

In the following sections, some general problems are solved and details of peculiar features of each problem are discussed.

5. Problems for which $Q(x) \gg 0$

This type of problem occurs when one investigates a Sturm–Liouville system exhibiting rapid oscillations. If there are sufficiently many oscillations to ensure that

$$\lambda\rho(x) - q(x) \gg 0, \qquad \forall x \in [a, b],$$

the modified Prüfer transformation is applied to (2.2) and the boundary conditions are transformed to the modified Prüfer phase. From property (b) of regular Sturm–Liouville systems, it is clear that, as $q(x)$ is bounded on $[a, b]$, such systems possess an infinite number of eigenvalues that may be determined using this approach.

The initial value problem is

$$\begin{gathered} \frac{d\phi}{dx} = [\lambda\rho(x) - q(x)]^{1/2} + \frac{\lambda\rho'(x) - q'(x)}{4(\lambda\rho(x) - q(x))} \sin 2\phi, \\ \phi(a) = \tilde{\alpha}, \end{gathered} \tag{5.1}$$

where $\tilde{\alpha}$ is given by (3.8a).

In this case, one defines f_m in terms of the modified Prüfer phase

$$f_m(\lambda) = \phi(b; \lambda) - (\tilde{\beta} + m\pi),$$

where $\tilde{\beta}$ is given by (3.8b).

For this problem, it is possible to define a second-order iteration by choosing an initial value λ_0 and using the result

$$f_m'(\lambda) \simeq (1/2\lambda^{1/2}) \int_a^b [\rho(t)]^{1/2}\, dt, \tag{5.2}$$

valid for large values of λ. This result is obtained by integrating

$$d\phi/dx \simeq [\lambda\rho(x)]^{1/2},$$

with respect to x, and then differentiating with respect to λ. Convergence for this case is faster than for any of the other cases considered here.

If, as in many problems of this type, $\rho(x) \equiv c$, where c is a constant, one may choose the step-size h to be larger than is usual for initial value problems, while the results remain as accurate.

The results given in this section may be compared with results obtained by asymptotic methods applied to Sturm–Liouville systems. The asymptotic methods often involve tedious calculations to obtain high-order accuracy, which may be achieved from the modified Prüfer method by altering the step-size. For moderately large values of λ, many terms of the asymptotic series are required, and the method used here is more efficient.

Numerical methods are also available for this type of problem. However, only Rayleigh–Ritz methods appear to be of comparable accuracy for moderately large values of the eigenvalue λ. For highly oscillatory solutions, even Rayleigh–Ritz methods lose accuracy, while the method used here retains the same number of decimal places as λ increases.

As an illustration of the above remarks, eigenvalues of Sturm–Liouville systems computed by means of (5.1) and (5.2), are presented in Tables I, II, and III. The first two tables indicate results for eigenvalue problems associated with Weber's equation

$$w'' + \{\lambda - x^2\}\, w = 0. \tag{5.3}$$

Two problems have been considered,

$$\begin{aligned} &\text{(i)} \quad w(0) = w(1) = 0, \\ &\text{(ii)} \quad w(0) = w'(1) = 0. \end{aligned} \tag{5.4}$$

In these tables, columns (a) and (b) are the eigenvalues calculated using 100 and 10 steps, respectively, in the Runge–Kutta method. The run time for the 20 eigenvalues of column (a) was approximately 30 sec, while for column (b), it was less than 4 sec. In Table I, results are presented for boundary condition (i).

TABLE I

First 20 Eigenvalues for Weber's Equation with Boundary Condition (5.4) (i)

	λ_m	
m	(a)	(b)
1	10.1511640305	10.1511640305
2	39.7993930037	39.79939302
3	89.1543424562	89.15434255
4	158.2439617071	158.2439620
5	247.0715002280	247.071501
6	355.6377438064	355.637746
7	483.9429592801	483.942965
8	631.9872575754	631.98727
9	799.7706915319	799.77074
10	987.2932889272	987.29326
11	1194.5550655004	1194.5549
12	1421.5560307115	1421.5559
13	1668.2961905247	1668.2961
14	1934.7755488515	1934.7754
15	2220.9941083433	2220.9940
16	2526.9520600122	2526.9518
17	2852.6488376921	2852.6486
18	3198.0850098354	3198.0847
19	3563.2603879971	3563.2597
20	3948.1749727202	3948.1751

Comparing the results of (i) with those of Birkhoff and Fix [4], it may be seen that column (a) of Table I appears to be accurate to 14 significant figures, while column (b) is accurate to 12 figures initially, leveling out after about 10 eigenvalues to six significant figure accuracy. To confirm whether or not this six figure accuracy is maintained for higher eigenvalue, problem (ii) was investigated for such eigenvalues. The results may be seen in Table II.

Asymptotic formulas for this problem show that the results in Table II column (a) are accurate to 10 decimal places. As suspected from the results of Table I, the error when 10 steps are used in the Runge–Kutta procedure, appears to remain less than 10^{-3}. The computation time per eigenvalues is less than in Table I as the relation (5.2) leads to a rapidly convergent iteration, often only one step being necessary.

It may be that the optimum choice of the number of steps for this problem is between 10 and 100. Such a choice would give an acceptable accuracy, while taking a small computation time. One would think that 30 or 40 steps would give a combination of the above properties.

TABLE II

Eigenvalues Corresponding to Highly Oscillatory Eigenfunctions of Weber's Equation Satisfying Boundary Condition (5.4) (ii)

	λ_m	
m	(a)	(b)
100	97711.8842956852	97711.8846
200	392813.0560529779	392813.0561
300	885306.3153903790	885306.3152
400	1575191.6632255727	1575191.6632
500	2462469.0988021103	2462469.0989
600	3547138.6223169501	3547138.6223
700	4829200.2341860869	4829200.2342
800	6308653.9339494741	6308653.9340
900	7985499.7216152999	7985499.7215
1000	9859737.5974535981	9859737.5975

This theme is pursued in a further example of this type, derived from Mathieu's equation. Several authors have computed the eigenvalues of this problem. In Table III, the results obtained by the method of this section are given.

Mathieu's equation

$$w'' + \{\lambda - 2q \cos 2x\} w = 0, \tag{5.5}$$

defines a regular Sturm–Liouville system on the interval $[0, \pi/2]$.

The number of eigenvalues that may be computed will depend on q as λ may be less than $2 \mid q \mid$. The eigenvalues for which $\lambda < 2 \mid q \mid$ may be computed using a combination of the Prüfer and modified Prüfer equations, as in Section 6. The Sturm–Liouville system whose eigenvalues have been calculated is Eq. (5.5) with the boundary condition

$$w(0) = w(\pi/2) = 0. \tag{5.6}$$

TABLE III

Some Eigenvalues of Mathieu's Equation Corresponding to Odd Eigenfunctions

	$(q = 1)$	$(q = 10)$	$(q = 25)$	$(q = 100)$
m	λ_m	λ_m	λ_m	λ_m
5	100.005050675	100.5067695	Note[a]	Note[a]
10	400.001253135	400.1253382	400.78419	Note[a]
15	900.00055617	900.0556195	900.34769	900.5836
100	40000.0000	40000.00	40000.0	40000.1
1000	4000000.000000	4000000.0000	4000000.000	4000000.00

[a] Not applicable.

In Table III 50 steps were used. The results given agreed with the results for 100 steps. The computing time for each eigenvalue was approximately 1 sec.

The higher eigenvalues in Table III may be checked against the first two terms of the asymptotic series for λ, namely,

$$\lambda_n = 4n^2 + (q^2/8n^2) + O(1/n^3).$$

Eigenvalues for $q = 1$ may be compared with the results of Birkhoff and Fix [4].

6. Positive Turning Point Problems

In this and the following section, turning-points of Sturm–Liouville systems are discussed. The class of turning point problems covered in this section is more adaptable to the present technique although it is the more difficult to treat by analytic methods.

The division of turning point problems into positive and negative types corresponds to the two cases of rapidly oscillating eigenfunctions and exponentially decaying eigenfunctions, respectively. Thus, a positive turning point problem may be defined as a Sturm–Liouville system for which

(i) $\exists$ a set $\{x_1, \ldots, x_k\} \subset [a, b]$ with k finite, such that $\lambda\rho(x_i) = q(x_i)$, $i = 1, 2, \ldots, k$,

(ii) If $\{\lambda\rho(x) - q(x)\} < 0$, for fixed λ, on a subinterval (a_i, b_i) of $[a, b]$ then

$$b_i - a_i \leqslant \left(\left\{\frac{1}{\inf_{x\in(a_i,b_i)}\{\lambda\rho(x) - q(x)\}}\right\}^{1/2}\right),$$

(iii) $\{\lambda\rho(x) - q(x)\} \gg 0$ on some interval of finite length.

Any other type of turning point will be a negative turning point. Examples of equations having positive turning points are

$$w'' + (\lambda x^2 - 1)\, w = 0, \qquad \text{for large } \lambda,\ x \in [0, 1]$$

or

$$w'' + \{\mu - 4q\cos^2 x\}\, w = 0, \qquad \text{for } \mu \simeq 4q \gg 0,\ x \in [0, \pi/2].$$

For this type of problem, a combination of Prüfer and modified Prüfer substitutions may be used to obtain the solution. In the regions for which $Q(x) < K$, the Prüfer substitution is used, and for those in which $Q(x) \geqslant K$, the modified Prüfer substitution is applied. Conversion between the two methods is facilitated by the results of Section 3. The value of the positive constant K is arbitrary, but one would normally choose $K = 1$.

As in the previous section, it is possible to define a second-order iteration to determine the zero of the function $f_m(\lambda)$. The value of θ changes slowly when $\{\lambda\rho(x) - q(x)\} = O(1)$, but if this function is large, $\partial\theta/\partial\lambda \doteq \rho(x)\sin^2\theta$. Thus, an analogous formula to (5.2) may be established for this type of problem, namely,

$$f_m'(\lambda) \doteq \frac{1}{2}\int_I \frac{\rho(t)}{[\lambda\rho(t) - q(t)]^{1/2}}\,dt,$$

where $I = \{x: x \in [a, b],\ \lambda\rho(x) \geqslant q(x) + 1\}$.

Three examples of this type of problem are given below. Each example is derived from a separation of variables technique applied to either Laplace's equation, or the reduced wave equation. Asymptotic methods may be used to determine implicit formulae for the eigenvalues (e.g., for parabolic cylinder functions see [12]) so that Prüfer methods appear to be the simplest approach to the positive turning point problem.

For the first example, Mathieu's equation is used. The problem is defined by Eqs. (5.5) and (5.6). For $q \gg 0$, the maximum value of $2q\cos 2x$ is $2q$ so that a positive turning point problem exists if λ is approximately equal to $2q$. This turning point occurs at the origin and is of second order as $(d/dx)\{\lambda - 2q\cos 2x\}\,|_{x=0} = 0$. These eigenvalues will lead to solutions of the Sturm–Liouville system having approximately $2q^{1/2}/\pi$ zeros in the interval $(0, \pi/2)$. Results from this type of problem are given in Table IV.

TABLE IV

Some Eigenvalues of Mathieu's Equation Corresponding to Odd Eigenfunctions

(q = 100, 50 steps)		(q = 1000, 100 steps)		(q = 10,000, 200 steps)	
m	λ_m	m	λ_m	m	λ_m
8	276.72837	21	2090.7371	64	20068.869
9	340.00030	22	2224.4649	65	20402.006
10	412.79665	23	2373.8457	66	20762.004
11	494.49811			67	21149.104

The computation time per eigenvalue was approximately equal to $n/50$ sec, where n is the number of steps. The results above agree with results obtained using 1000 steps.

The comparison equation in turning point analysis for a differential equation possessing a positive turning point of order two is the parabolic cylinder equation

$$w'' + \{\lambda + \gamma^2 x^2\}\,w(x) = 0. \tag{6.1}$$

The eigenvalues of this equation for all types of boundary condition are particularly important. Sleeman in [12] obtained a relation between λ and the zeros of both even and odd solutions of (6.1). The positive turning point case corresponds to large values of γ and both small values of λ and values of $O(\gamma)$. Such values of λ and the number of the eigenvalue are given in Table V for various values of γ, for the problem defined by (6.1) and the boundary condition

$$w(0) = w(1) = 0.$$

TABLE V

Some Eigenvalues of the Parabolic Cylinder Equation Corresponding to Odd Eigenfunctions

($\gamma = 50$, 100 steps)		($\gamma = 100$, 100 steps)		($\gamma = 200$, 200 steps)	
m	λ_m	m	λ_m	m	λ_m
9	121.0785	17	219.893	32	17.4443
10	279.0427	18	483.351	33	424.0872
11	465.1663				

The details of the computation are the same as those for Table IV.

A third problem both of physical interest yet having known solution in some cases is that of the Lamé polynomials. In [7], a relation has been given for the eigenvalue h of Lamé's equation

$$w'' + \{h - n(n+1)\, k^2 sn^2 z\}\, w = 0, \tag{6.2}$$

snz being the Jacobian elliptic function of modulus k, when n is large. To obtain Lamé polynomials as a solution of (6.2), n must be an integer. If $k^2 = 0.5$, then in certain cases, the eigenvalue h is equal to $\frac{1}{2}n(n+1)$. These cases are as follows,

(i) $n = 4m$, $w'(0) = w'(K) = 0$, $(m+1)$th eigenvalue,
(ii) $n = 4m + 1$, $w'(0) = w(K) = 0$, $(m+1)$th eigenvalue,
(iii) $n = 4m + 2$, $w(0) = w'(K) = 0$, $(m+1)$th eigenvalue,
(iv) $n = 4m + 3$, $w(0) = w(K) = 0$, $(m+1)$th eigenvalue.

In the above, K is complete elliptic integral of the first kind. For large values of n, the results obtained by Prüfer methods appear to be accurate to 10 significant figures using 100 steps in the Runge–Kutta technique. The computation time for each eigenvalue is a little higher than in Table IV, as elliptic functions have to be evaluated. However, if several eigenvalues are calculated without changing k, then these values may be stored and the computation time is not significantly affected.

The method proposed here provides an apportunity to verify some of the results of [7] when $k^2 \neq 0.5$. For example, if $k^2 = 0.25$, then one would expect that the $(m + 1)$th eigenvalue of the problem

$$n = 6m, \qquad w'(0) = w'(K) = 0,$$

is approximately equal to $\frac{1}{4}n(n + 1)$. Typical results for this problem are given in Table VI.

TABLE VI

Eigenvalues of Lamé's Equation Corresponding to Even Lamé Polynomials with Modulus 0.25

n	h
60	916.49
120	3632.80
180	8149.05
900	202742.6

The computation time for this problem involving 100 steps is as given in Table VI.

These results agree with the theoretical result [7].

7. Other Problems

More general turning point problems will have either two regions on each side of the turning point in which $\{\lambda\rho(x) - q(x)\}$ is negative or a side for which $\{\lambda\rho(x) - q(x)\}$ is positive and one for which it is negative. When either of these conditions occurs, we lose our estimates of the type (5.2) for $f_m'(\lambda)$, and consequently, one should expect the method not to be as efficient for this type of problem.

In Tables VII and VIII, one example of a turning point problem for moderately large negative values of $(\lambda\rho - q)$ is given. The second example is a typical example of a regular Sturm–Liouville problem, in which all functions are of order one.

Mathieu's equation again provides an example, this time of a general turning point problem. The results given in Table VII may be compared with the results of [1]. The effects of having a finite region of absolute stability may be seen in this problem defined by Eq. (5.5) with boundary condition

$$w'(0) = w'(\pi/2) = 0. \tag{7.2}$$

TABLE VII

Some Eigenvalues of Mathieu's Equation
Corresponding to Even Eigenfunctions

	λ_1		λ_2		λ_6	
q	(i)	(ii)	(i)	(ii)	(i)	(ii)
5	−5.800046	−5.800046	7.449110	7.449110	100.126369	100.126369
10	−13.936980	−13.936980	7.717367	7.717370	100.506770	100.506770
15	−22.513037	−22.513038	5.077962	5.077982	101.145203	101.145203
20	−31.313419	−31.313391	1.154094	1.154275	102.048916	102.048916
25	−40.256785	−40.256783	−3.522105	−3.522190	103.230205	103.230205

Column (i) of Table VII was obtained using 50 steps, while column (ii) was obtained using 100 steps. Computation time per eigenvalue was 3 sec for column (i) and 6 sec for column (ii). Fifty is almost the smallest possible choice for the number of steps, and one may compare the results of columns (i) and (ii) for the first and sixth eigenvalues. In the former case, $\{\lambda - 2q \cos 2x\}$ takes large negative values, while in the latter case, this expression is large and positive. The results for columns (i) and (ii) for the first eigenvalue, indicate that 50 steps leads to answers accurate to six significant figures, while the sixth eigenvalue has column (i) accurate to nine significant figures.

Finally, in Table VIII, the first few eigenvalues of

$$w'' + \{\lambda x - x^4\}\, w(x) = 0, \tag{7.3a}$$

$$w'(1) - w(1) = w'(2) - 4w(2) = 0, \tag{7.3b}$$

are computed.

TABLE VIII

First Five Eigenvalues for the Problem
Defined by Eqs. (7.3)

n	λ_n
1	2.00000000
2	13.47421073
3	33.63786454
4	66.86980203
5	113.37160420

Using 100 steps, the computing time for an eigenvalue was approximately 5 sec. The first of the results in Table VIII may be seen to agree with the theoretical value to eight decimal places.

8. Concluding Remarks

It has been shown in the earlier sections that the Prüfer transformations lead to an efficient technique for numerical solution of regular Sturm–Liouville sytems. For problems in which only small numbers are involved, the Prüfer technique appears to be as efficient as the Rayleigh–Ritz method and more efficient than all other methods. However, the Prüfer techniques become more efficient for successive eigenvalues, while the Rayleigh–Ritz method being an algebraic method becomes less efficient.

The Prüfer techniques used here seem readily adaptable to a problem related to the eigenvalue problem, namely, the problem of determing the zeros of the solution of a given differential equation.

It would also be interesting to consider singular Sturm–Liouville problems on the halfline using the Prüfer technique combined with an approximate value of the phase, obtained by the W.K.B. method at a finite distance on the real line.

A third extension of the Prüfer technique concerns the case in which $q(x)$ is a step function or a piecewise continuous function. A generalization of the ideas of the Prüfer substitution will allow this case to be treated.

However, the author feels the most interesting aspect of Prüfer methods is that they may be applied immediately to the multiparameter problem. From the examples given here, it appears that Prüfer methods will be sufficiently fast to enable one to solve multiparameter problems containing more than two parameters in a reasonable amount of computing time.

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